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Existence of Higher Dimensional Invariant Tori for Hamiltonian Systems

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This paper is on the persistence of higher-dimensional invariant tori for Hamiltonian systems. A KAM type theorem is established. © 1998 Academic Press

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1. INTRODUCTION

The celebrated KAM theory says that under a certain nondegeneracy condition, a nearly integrable Hamiltonian system preserves a family of invariant tori [1]. Recently this theory has undergone some considerable developments [2–10]. However, the study of the existence of higher dimensional invariant tori (KAM tori) for Hamiltonian systems seems quite rare. Parasyuk [7] and Hermann [13] made interesting observations on such problems. In the present paper, motivated by their work, we shall consider more general situations.

Now let us consider a Hamiltonian function,

$$H(x, y) = N(y) + P(x, y), \tag{1.1}$$

where N is a real analytic function defined on some closed bounded and connected region $G \subset R^l$; P is a real analytic function defined on $T^m \times G$. Here $m + l$ is even and $m \geq l$; $T^m = R^m/2\pi Z^m$ denotes the torus of dimension m.

Let $(T^m \times G, \omega^2)$ be a symplectic manifold, and I an analytic Hamiltonian homeomorphism from the 1-form space to vector fields, that is, let I be an antisymmetric matrix such that the relation $\omega^2(\cdot, I\omega^1) = \omega^1(\cdot)$, for all 1-form ω^1 defined on $T^m \times G$ (see [11]). Then the 2-form ω^2 can be

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determined in the following way: $\{f_1, f_2\} = df_1(Idf_2) = \omega^2(Idf_1, Idf_2)$, for all smooth functions f_1 and f_2 defined on $T^m \times G$, where $\{\cdot, \cdot\}$ denotes the usual Poisson bracket.

Let ω^2 be invariant relative to T^m. Thus the coefficients of ω^2 and the matrix I are independent of the coordinate x. Hence the Hamiltonian system corresponding to (1.1) has the following form [12]:

$$\dot{z} = I(y)\mathrm{grad}^T H(z), \tag{1.2}$$

where $z = (x, y)$, and the superscript T denotes the transpose of a vector. By Lemma 1 (see Appendix), we have that

$$I(y)\mathrm{grad}^T N(y) = \Big(\omega(y), \underbrace{0, \dots, 0}_{l}\Big)^T.$$

Throughout this paper, we make the following hypotheses:

$$\mathrm{rank}\left(\frac{\partial \omega}{\partial y}\right) = r, \qquad \text{on } G, \tag{1.3}$$

$$\mathrm{rank}\left\{\omega, \frac{\partial^\alpha \omega}{\partial y^\alpha} : \forall \alpha \in Z_+^l, 0 < |\alpha| \le m - r + 1\right\} = m, \qquad \text{on } G, \tag{1.4}$$

where Z_+ stands for the set of nonnegative integers, and $\partial^\alpha \omega / \partial y^\alpha = (\partial^\alpha \omega_1 / \partial y^\alpha, \dots, \partial^\alpha \omega_m / \partial y^\alpha)^T$.

Set

$$G_\rho = \{y \colon \mathrm{Re}\, y \in G, |\mathrm{Im}\, y| < \rho\},$$

$$\Sigma_\rho = \{x \colon \mathrm{Re}\, x \in T^m, |\mathrm{Im}\, x| < \rho\}.$$

In what follows, for a vector, $|\cdot|$ denotes its maximum norm in components; for a function, $|\cdot|$ denotes the usual supremum norm; $\langle \cdot, \cdot \rangle$ denotes the usual inner product in the corresponding Euclidean spaces.

Now we are in the position to state our main result:

THEOREM A. *Assume that $H(x, y)$ is a real analytic function on $\Sigma_\rho \times G_\rho$, and $N(y)$ satisfies the conditions* (1.3) *and* (1.4) *on G. Then there exist $\varepsilon_0 > 0$ and a nonempty Cantor set $G_{\delta_0} \subset G$ such that $\forall \varepsilon > 0$, whenever $|P| \le \varepsilon < \varepsilon_0$,* (1.2) *preserves a family of invariant tori IT_y, $y \in G_{\delta_0}$, whose frequency $\omega^\infty(y)$ satisfies*

$$|\omega^\infty - \omega| \le c\varepsilon_0^{3/4},$$

and the following measure estimate holds:

$$\mathrm{mes}_l\big(G \setminus G_{\delta_0}\big) \le c\delta_0^{1/(m-r+1)},$$

where c is a constant independent of ε *and* δ_0, δ_0 *depends only on* ε_0, *and* $\delta_0 \to 0$ *as* $\varepsilon_0 \to 0$.

Remark 1. In [7], a nondegeneracy condition on $\omega(y)$ is needed, and that condition does not seem obvious. But in our result, the conditions (1.3) and (1.4) are obvious, and as $r < m$, the unperturbed system possesses a certain degeneracy.

Remark 2. If $m = l = r$, and

$$I(y) = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix} \qquad (E_m \text{ is the unit matrix of order } m),$$

then Theorem A is the classical KAM theorem.

Remark 3. The discussion of the case $m < l$ is similar. The only difference is the measure estimate of invariant tori, which can be completed in a certain way (see [10]).

2. A SMALL DIVISOR PROBLEM

In this section we shall give an auxiliary result that deals with a small divisor problem. Throughout the paper we also use the following notations.

Set $y_0 \in G$. Write

$$D(\rho, s) = \{(x, y): \operatorname{Re} x \in T^m, |\operatorname{Im} x| < \rho, |y - y_0| < s^2\},$$

$$[P](y) = \frac{1}{(2\pi)^m} \int_{T^m} P(x, y)\, dx.$$

Thus if

$$P(x, y) = \sum_{k \in Z^m} P_k e^{\sqrt{-1}\langle k, x\rangle},$$

then $[P] = P_0$. We shall use c_i's to denote some positive constants depending only on the constants ε_{00}, Θ, τ, M, m, l, and ρ, where ε_{00}, Θ, τ, and M will be given in the sequel.

Let F and N be real analytic functions defined on $D(\rho, s)$, and

$$I(y_0)\operatorname{grad}^T N(y_0) = (\omega, 0),$$

which satisfies

$$|\langle k, \omega\rangle| \geq \delta |k|^{-r}, \qquad 0 \neq k \in Z^m,$$

where $\tau \geq m(m - r + 1) - 1$ is constant, $\delta > 0$, and $|k| = |k_1| + \cdots + |k_m|$.

Set

$$N_1 = \left\langle \operatorname{grad}^T N(y_0), y - y_0 \right\rangle.$$

We have the following.

PROPOSITION. *The equation*

$$\{\varphi, N_1\} = F - [F], [\varphi] = 0 \tag{2.1}$$

has a unique real analytic solution φ, *and*

$$|\varphi|_{D(\rho - \delta, s)} \leq \frac{c_1}{\delta^{m+r+2}} |F|_{D(\rho, s)}.$$

Proof. Write $D = D(\rho, s)$. Clearly,

$$\{\varphi, N_1\} = d\varphi(IdN_1) = \langle \omega, \operatorname{grad}^T \varphi \rangle.$$

Let

$$F = \sum_{k \in Z^m} F_k(y) e^{\sqrt{-1}\langle k, x \rangle}. \tag{2.2}$$

Putting (2.2) into (2.1) yields

$$\varphi = \sum_{k \in Z^m \setminus 0} \frac{F_k(y)}{\langle k, \omega \rangle} e^{\sqrt{-1}\langle k, x \rangle}.$$

By Cauchy's estimate we have

$$\begin{aligned}
|\varphi|_{D(\rho - \delta, s)} &\leq \sum_{k \in Z^m \setminus 0} \frac{|F_k(y)|}{|\langle k, \omega \rangle|} e^{|k|(\rho - \delta)} \\
&\leq \sum_{k \in Z^m \setminus 0} \frac{|k|^\tau}{e^{|k|\delta}\delta} |F|_D \\
&\leq \sum_{j=1}^{\infty} \frac{2^m j^{m+\tau-1}}{\delta e^{j\delta}} |F|_D \qquad \text{(see [1])} \\
&\leq \frac{c_2}{\delta^{m+\tau+2}} \sum_{j=1}^{\infty} \frac{1}{j^{2+[\tau]-\tau}} |F|_D \qquad ([\tau] \text{ is the integer part of } \tau) \\
&\leq \frac{c_1}{\delta^{m+\tau+2}} |F|_{D(\rho, s)}.
\end{aligned}$$

3. PROOF OF THEOREM A

Fix $y_0 \in G$, and write

$$N = N(y_0) + \left\langle \operatorname{grad}^T N(y_0), y - y_0 \right\rangle + \frac{1}{2!}\left\langle \frac{\partial^2 N}{\partial y^2}(y_t)(y - y_0), y - y_0 \right\rangle$$

$$= N_0 + N_1 + \hat{N}, \tag{3.1}$$

where $y_t = y_0 + t(y - y_0)$, $0 \le t \le 1$.

Remark 4. In general, we may take

$$N = N_0 + N_1(y) + \hat{N}(x, y),$$

where $\hat{N}(x, y) = \langle Q(x, y)(y - y_0), y_0 - y \rangle$. By (1.2) it is easy to see that $\{(x, y)\colon x \in T^m,\ y = y_0\}$ is an invariant torus of dimension m of Hamiltonian N.

Since $\operatorname{grad}^T N(y)$ and $\partial^2 N/\partial y^2$ are bounded on G, we may assume that on $D(\rho, s')$,

$$|N_1| \le \mu_1^0 (s')^2, \quad |\hat{N}| \le \mu_2^0 (s')^4, \quad s' \le s, \tag{3.2}$$

where μ_1^0 and μ_2^0 are constants depending on G.

1. *Outline of the proof and estimate of the measure*

Choose convergent sequences:

$$\varepsilon_{i+1} = \varepsilon_i^{9/8}, \quad \delta_i = \varepsilon_i^{1/8(m+\tau+3)}, \quad s_i = \varepsilon_i^{1/8},$$

$$\rho_{i+1} = \rho_i - 6\delta_i, \quad i = 0, 1, 2, \dots,$$

where constant ε_0 satisfies conditions (A)–(H) listed below, and $\rho_0 = \rho$.

Set

$$D_j^i = D\left(\rho_i - j\delta_i, \frac{7-j}{8}s_i\right)$$

$$= \left\{(x, y)\colon \operatorname{Re} x \in T^n, |\operatorname{Im} x| < \rho_i - j\delta_i, |y| < \left(\frac{7-j}{8}s_i\right)^2\right\},$$

$$0 \le j \le 6.$$

Assume that the Hamiltonian function $N + P$ on D_0^i is, by a symplectic transformation, reduced into $N^i + P^i$, where

$$N^i = N_0^i + N_1^i + \hat{N}^i, \tag*{$(3.3)_i$}$$

$$|N_1^i|_{D(\rho_i, s')} \le \mu_1^i (s')^2, \quad |\hat{N}^i|_{D(\rho_i, s')} \le \mu_2^i (s')^4, \quad s' \le s_i \tag*{$(3.4)_i$}$$

$$|P^i| \le \varepsilon_i. \tag*{$(3.5)_i$}$$

Set

$$O_i = \left\{ y \in G: \left| \left\langle k, \omega^i(y) \right\rangle \right| \geq \delta_i |k|^{-r}, 0 \neq k \in Z^m \right\}.$$

Take $y_0 \in O_i$. Then

$$I(y_0) \mathrm{grad}^T N_1^i(y_0) = \left(\omega^i(y_0), 0 \right).$$

Introduce a symplectic transformation Φ_i such that

$$(N^i + P^i) \circ \Phi_i = N^{i+1} + P^{i+1}.$$

We are going to prove that $N^{i+1} + P^{i+1}$ satisfies the corresponding expressions $(3.3)_{i+1}$–$(3.5)_{i+1}$. If

$$y_0 \in \bigcap_{k=0}^{i+1} O_k \neq \phi,$$

then the iteration processes can continue. Set

$$\begin{aligned} N^{i+1} &= N^i + R^i \\ &= \left(N_0^i + [P^i](y_0) \right) + \left(N_1^i(y) + \left\langle \frac{\partial}{\partial y} [P^i](y_0), y - y_0 \right\rangle \right) \\ &\quad + \left(\hat{N}^i(x, y) + \hat{R}^i(x, y) \right) \\ &= N_0^{i+1} + N_1^{i+1} + \hat{N}^{i+1}, \end{aligned} \tag{3.6$_i$}$$

where R^i is determined by the following (3.11). Without loss of generality, we assume

$$\Theta \leq |I|_G \leq \Theta^{-1}, \tag{3.7}$$

where

$$|I|_G = \max_{y \in G} \sup_{|z|=1} \frac{|Iz|}{|z|}, \qquad z = (x, y).$$

From $(3.5)_i$ and Cauchy's estimate, we derive

$$\left| \mathrm{grad}^T [P^i] \right|_{D_0^i} \leq \frac{c_3}{s_i^2} \varepsilon_i^{3/4}. \tag{3.8}$$

By (3.6)–(3.8), we see that as

$$\varepsilon_0 < \min \left\{ \left(\frac{1}{2} \right)^{32}, \frac{\varepsilon_{00}}{2 c_3} \Theta \right\}, \tag{A}$$

the following inequality holds on D_1^{i+1}:

$$
\begin{aligned}
|\omega^{i+1} - \omega^0| &\le \sum_{k=0}^{i} |\omega^{k+1} - \omega^k|_{D_1^k} \\
&\le \sum_{k=0}^{i} |I|_G \left| \operatorname{grad}^T [P^{k+1}] \right|_{D_1^k} \\
&\le c_3 \Theta^{-1} \sum_{k=0}^{i} \varepsilon_k^{3/4} < \varepsilon_{00},
\end{aligned}
$$

where $\omega^0 = \omega$, and ε_{00} is given in the Appendix. Hence

$$
|\omega^{i+1}| \le |I|_G |\operatorname{grad}^T N|_G + \varepsilon_{00} \le M + \varepsilon_{00}, \tag{3.10}
$$

where $M \ge \max\{|I|_G |\operatorname{grad}^T N|_G,\ \mu_2^0 + 2c_{15},\ \mu_1^0 + 2c_{14}\}$, and c_{14} and c_{15} will be determined below.

Set

$$
G_{\delta_0} = \bigcap_{i=0}^{\infty} O_i .
$$

By (3.9) and Lemma 3 we obtain

$$
\begin{aligned}
\operatorname{mes}_l\left(G \setminus G_{\delta_0}\right) &\le \sum_{i=0}^{\infty} \operatorname{mes}_l(G \setminus O_i) \\
&\le \sum_{i=0}^{\infty} \operatorname{mes}_l(G \setminus O_i) \\
&\le c \sum_{i=0}^{\infty} \delta_i^{1/(m-r+1)} \\
&\le 2c\,\delta_0^{1/(m-r+1)} .
\end{aligned}
$$

Here we have used the inequality

$$
\varepsilon_0 < \left(\frac{1}{2}\right)^{64(m-r+1)(m+r+3)} . \tag{B}
$$

2. *KAM iteration*

We prove only one cycle of iteration processes, to say, from the ith step to the $i+1$th step. For simplicity, we omit notation i and denote $i+1$ by $+$.

We need to construct functions $S = S_0 + S_1$, $R = R_0 + R_1 + \hat{R}$ such that

$$\{S, N\} = P - R, \qquad [S] = 0, \qquad [R_0 + R_1] = R_0 + R_1. \quad (3.11)$$

Let Φ^t be the flow determined by the vector field $I\,dS$. Define $\Phi = \Phi^1$. Then Φ is a symplectic transformation. On the basis of the fact

$$\frac{d}{dt} F \circ \Phi^t = \{F, S\} \circ \Phi^t$$

and Taylor's formula, we have

$$(N + P) \circ \Phi = N^+ + \int_0^1 \{tP + (1-t)R, S\} \circ \Phi^t \, dt, \qquad (3.12)$$

$$N^+ = N + R, \qquad P^+ = \int_0^1 \{tP + (1-t)R, S\} \circ \Phi^t \, dt. \qquad (3.13)$$

Using (3.11) yields

$$\{S_0, N_1\} = P(x, y_0) - R_0, \qquad [S_0] = 0, \quad [R_0] = R_0, \qquad (3.14)$$

$$\{S_1, N_1\} = \left\langle \frac{\partial P}{\partial y}(x, y_0), y - y_0 \right\rangle - \left\{ S_0, \frac{1}{2} \left\langle \frac{\partial^2 N}{\partial y^2}(y_0)(y - y_0), y - y_0 \right\rangle \right\}$$
$$-R_1, \qquad [S_1] = 0, \quad [R_1] = R_1. \qquad (3.15)$$

Take

$$R_0 = [P](y_0) \qquad R_1 = \left\langle \left[\frac{\partial P}{\partial y} \right](y_0), y - y_0 \right\rangle, \qquad (3.16)$$

$$\hat{R} = \hat{P} - \left\{ S_0, \hat{N} - \frac{1}{2} \left\langle \frac{\partial^2 N}{\partial y^2}(y_0)(y - y_0), y - y_0 \right\rangle \right\} - \{S_1, \hat{N}\}, \quad (3.17)$$

$$\hat{P} = P - P(x, y_0) - \left\langle \frac{\partial P}{\partial y}(x, y_0), y - y_0 \right\rangle. \qquad (3.18)$$

Utilizing (3.5), (3.14), (3.16), and the proposition, we have

$$|S_0|_{D_1} \leq \frac{c_4}{\delta^{m+r+2}} \varepsilon. \qquad (3.19)$$

By (3.16), (3.19), and Cauchy's estimate, we obtain

$$\begin{aligned}
&\left|\left\langle \frac{\partial P}{\partial y}(x, y_0), y - y_0 \right\rangle - \left\{ S_0, \frac{1}{2}\left\langle \frac{\partial^2 N}{\partial y^2}(y_0)(y-y_0), y - y_0 \right\rangle \right\} - R_1 \right|_{D_2} \\
&\quad \leq 2\left|\left\langle \frac{\partial P}{\partial y}(x, y_0), y - y_0 \right\rangle\right|_{D_2} \\
&\qquad + \left|\left\{ S_0, \frac{1}{2}\left\langle \frac{\partial^2 N}{\partial y^2}(y_0)(y-y_0), y - y_0 \right\rangle \right\}\right|_{D_2} \\
&\quad \leq c_4 \varepsilon + \frac{c_1 \varepsilon}{\delta^{m+r+3}} \cdot \mu_2 |I|_G \\
&\quad \leq \frac{c_5}{\delta^{m+r+3}} \varepsilon \big(c_5 = M\Theta^{-1}(c_1 + c_4),\ \mu_2^0 + 2c_{15} \leq M \big).
\end{aligned} \tag{3.20}$$

Here we have used the inequality $\mu_2 \leq \mu_2^0 + 2c_{15}$! The proof will be given in the sequel. Applying the proposition yields

$$|S_1|_{D_3} \leq \frac{c_1}{\delta^{m+r+2}} \cdot \frac{c_5}{\delta^{m+r+3}} \varepsilon \leq \frac{c_6}{\delta^{2m+2\tau+5}} \varepsilon. \tag{3.21}$$

From (3.18) and (3.4) we see

$$|\hat{P}|_{D_1} \leq c_7 \varepsilon, \qquad \max\left\{ \left|\frac{\partial \hat{N}}{\partial y}\right|_{D_1}, \left|\frac{\partial^2 \hat{N}}{\partial y^2}\right|_{D_1} \right\} \leq c_8\, \mu_2 \leq c_8 M. \tag{3.22}$$

By (3.17), (3.19)–(3.22), (3.7), and Cauchy's estimate, we have

$$\begin{aligned}
|\hat{R}|_{D_4} &\leq c_7 \varepsilon + \frac{c_4 \varepsilon}{\delta^{m+r+3}} \cdot 2c_8 M |I|_G + \frac{c_6}{\delta^{2m+2\tau+6}} \varepsilon \cdot c_8 M |I|_G \\
&\leq \frac{c_9}{\delta^{2m+2\tau+6}} \varepsilon.
\end{aligned} \tag{3.23}$$

(3.19) and (3.21) imply

$$|S|_{D_3} \leq \frac{c_{10}}{\delta^{2n+2\tau+6}} \varepsilon. \tag{3.24}$$

From (3.13), (3.23), (3.24), and Cauchy's estimate, it follows that

$$\begin{aligned}
|P^+|_{D_4} &\leq |\{P, S\}|_{D_4} + |\{R, S\}|_{D_4} \\
&\leq \frac{c_{11}\Theta^{-1}}{s^2 \delta^{2m+2\tau+6}} \varepsilon^2 + \frac{c_{12}}{s^2 \delta^{4m+4\tau+12}} \varepsilon^2 \\
&\leq \frac{c_{13}}{s^2 \delta^{4m+4\tau+12}} \varepsilon^2.
\end{aligned}$$

By the choice of s and δ, we have

$$|P^+| \leq \frac{1}{2}\varepsilon_+ < \varepsilon_+, \tag{3.25}$$

provided

$$\varepsilon_0 < (2c_{13})^{-8}. \tag{C}$$

Now we verify $(3.4)_+$ for N^+. By $(3.6)_+$ we obtain

$$|N_1^+|_{D(\rho-6\delta, s')} \leq \left(\mu_1 + c_{14}\frac{\varepsilon}{s^2}\right)(s')^2 \leq \mu_1^+(s')^2, \qquad s' \leq \frac{1}{8}s,$$

$$|\hat{N}^+|_{D(\rho-6\delta, s')} \leq \left(\mu_2 + c_{15}\frac{\varepsilon}{s^4}\right)(s')^4 \leq \mu_2^+(s')^4, \qquad s' \leq \frac{1}{8}s.$$

As

$$\varepsilon_0 < \left(\frac{1}{8}\right)^{64}, \tag{D}$$

we also have

$$s_+ < \frac{1}{8}s,$$

$$\mu_1 \leq \mu_1^0 + 2c_{14} \leq M,$$

$$\mu_2 \leq \mu_2^0 + 2c_{15} \leq M.$$

These imply $(3.4)_+$.

3. *Estimates of coordinates*

By (3.24) and Cauchy's estimate we derive

$$\left|\frac{\partial S}{\partial x}\right|_{D_4} \leq \frac{c_{14}}{\delta^{2m+2\tau+6}}\varepsilon, \qquad \left|\frac{\partial S}{\partial y}\right|_{D_4} \leq \frac{c_{15}}{s^2\delta^{2m+2\tau+6}}\varepsilon.$$

Choosing

$$\varepsilon_0 < (c_{14} + c_{15})^{-8}, \tag{E}$$

we obtain

$$|\Phi - \mathrm{id}|_{D_5} \leq \frac{c_{14} + c_{15}}{s^2\delta^{2m+2\tau+6}}\varepsilon < \varepsilon^{3/8} < s_+ < \delta_+. \tag{3.26}$$

Hence Φ is well defined on $D(\rho_+, s_+)$ and

$$\Phi\colon D(\rho_+, s_+) \to D_5. \tag{3.27}$$

4. *Completion of the proof*

Choose $y_0 \in G_{\delta_0}$ and define $\Psi_i = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{i-1}$. Then as

$$\varepsilon_0 < \min\left\{\left(\frac{1}{2}\right)^{64(m+r+3)}, \left(\frac{\rho}{24}\right)^{8(m+r+3)}\right\}, \tag{F}$$

we have

$$\rho_\infty = \rho - 6\sum_{i=0}^{\infty} \delta_i > \frac{\rho}{2}.$$

Set

$$D_\infty = \{x\colon \operatorname{Re} x \in T^m, |\operatorname{Im} x| < \rho_\infty\} \times \{y = y_0\}.$$

Using (3.26) and (D) yields that on D_∞,

$$|\Psi_i - \mathrm{id}| \le \sum_{k=1}^{i} |\Psi_k - \Psi_{k-1}| \le \sum_{k=1}^{i} \varepsilon_i^{3/8} < 2\varepsilon_0^{3/8}.$$

Hence $\{\Psi_i\}$ is uniformly convergent on D_∞. As in (3.9), from (A) we see

$$|\omega^\infty(y) - \omega(y)| < c_{16}\varepsilon_0^{3/4}, \qquad y \in G_{\delta_0},$$

where $\omega^\infty = \lim_{i\to\infty} \omega^i(y)$. By (3.26) we have

$$\left|\frac{\partial \Phi_i}{\partial(x_i, y_i)} - E\right| \le \frac{c_{17}}{s_i^2 \delta_i} \varepsilon_i^{3/8} < \varepsilon_i^{1/24},$$

as

$$\varepsilon_0 < c_{17}^{-24}, \tag{G}$$

where E is the unity matrix. Thus

$$\left|\frac{\partial \Psi_i}{\partial(x_i, y_i)}\right|_{D_\infty} \le \sum_{k=0}^{i-1} \left(1 + \varepsilon_k^{1/24}\right) < \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{3}{2},$$

whenever

$$\varepsilon_0 < \left(\frac{1}{2}\right)^{132}. \tag{H}$$

Therefore $\{\partial \Psi_i / \partial(x_i, y_i)\}$ converges on D_∞.

By (3.15)–(3.16) and (3.23) we may assume that

$$\hat{R}^i(x, y) = O\big((y - y_0)^2\big)$$

uniformly holds on D_4^i. Hence the system

$$z = I(y)\mathrm{grad}^T N^{i+1}(z)$$

has an invariant torus $\{(x, y)\colon x \in T^m,\ y = y_0\}$. Using the convergence of $\{\Psi_i\}$ and $\{\partial\Psi_i/\partial(x_i, y_i)\}$, we derive that $H^\infty = H \circ \Psi^\infty = N^\infty$, $\Psi^\infty = \lim_{i\to\infty}\Psi^i$, has an invariant torus $\{(x, y)\colon x \in T^m,\ y = y_0\}$ with the frequency $\omega^\infty(y_0)$. This completes the proof.

4. APPENDIX

Now we list some lemmas that have been used in previous sections.

LEMMA 1 [7]. *Consider the Hamiltonian system*

$$\dot{z} = I(y)\mathrm{grad}^T H(z), \qquad z = (x, y). \tag{4.1}$$

If $H(z) = H(y)$, then (4.1) *has the form*

$$\dot{x} = \omega(y), \qquad \dot{y} = 0,$$

where $\omega(y) = (\omega_1(y), \dots, \omega_m(y))^T$.

LEMMA 2 [9]. *Let $O \subset R^n$ be a bounded and connected region, and let g and $\bar{g}$ be real analytic n-dimensional vector-valued functions, on $\overline{O}$, such that*

$$\mathrm{rank}\left(\frac{\partial g}{\partial y}\right) = r,$$

$$\mathrm{rank}\left\{g, \frac{\partial^\alpha g}{\partial y^\alpha} : \forall\alpha, |\alpha| \le n - r + 1\right\} = n;$$

then there is $\varepsilon_0 > 0$ such that as δ is sufficiently small and $|\bar{g}| < \varepsilon_0$, the set

$$O_\delta = \left\{y \in O\colon \left|\langle k, g(y) + \bar{g}(y)\rangle\right| \ge \delta|k|^{-\tau}, 0 \ne k \in Z^n\right\}$$

is a nonempty Cantor set, and

$$\mathrm{mes}_n(O \setminus O_\delta) \le c\delta^{1/(n-r+1)},$$

where c is independent of α, $\bar{g}$.

LEMMA 3. *Assume that* $\omega(y)$ *satisfies conditions* (1.3) *and* (1.4), *and* $\tilde{\omega}(y)$ *is a real analytic m-dimensional vector-valued function. Then there is* $\varepsilon_{00} > 0$ *such that as* δ *is sufficiently small and* $|\tilde{\omega}| < \varepsilon_{00}$, *the set*

$$G_\delta = \left\{ y \in G : \left| \langle k, \omega(y) + \tilde{\omega}(y) \rangle \right| \geq \delta |k|^{-\tau}, 0 \neq k \in Z^m \right\}$$

is a nonempty Cantor set, and

$$\operatorname{mes}_l(G \setminus G_\delta) \leq c\delta^{1/(m-r+1)},$$

where c is independent of α *and* $\tilde{\omega}$.

Proof. Apply Lemma 2 and Fubini's theorem.

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